Extremal and Probabilistic Graph Theory

March 17

- **Recall : Erdös-Stone Theorem.**

For $\forall$ graph $F$ with $\chi(F) \geq 2$, we have ex$(n, F)$= $(1-\frac{1}{\chi(F)}+\mathrm{o}(1))\binom{n}{2}$.

We consider an improvement by providing a quantitative bound.

- **Definition.** For $\forall\, r \geq 2$ and $\varepsilon > 0$, let

$$f_r(n,\varepsilon) = \max\{m : \mathrm{ex}(n, T_r(rm)) \leq \mathrm{ex}(n, K_r) + \varepsilon\binom{n}{2} - 1\}$$

- **Remark.**

(1) If $m$ increases, ex$(n, T_r(rm)$ will also increases.

(2) Erdős-Stone Theorem tells for fixed $m, \varepsilon > 0$, this inequality holds when $n \to \infty$. But now we consider the counterpart, that is, $n, \varepsilon$ fixed, how large $m$ can be.

The meanings of $f_r(n,\varepsilon) < C$ is ex$(n, T_r(rC)) >$ ex$(n, K_r)$ + $\varepsilon\binom{n}{2} - 1$.

- **Definition.** For $f(n), g(n)$,

(1) $f \sim g$ if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$ .

(2) $f \lesssim g$ if $\lim_{n\to\infty} \frac{f(n)}{g(n)} \leq 1$.

- **Theorem 1.** $f_2(n,\varepsilon) \gtrsim \log_{1/\varepsilon} n$.

- **Proof of Theorem 1.** Recall K-S-T Theorem,

ex$(n, T_2(2m))$ = ex$(n, K_{m,m}) \leq \frac{1}{2}(m-1)^{\frac{1}{m}} n^{2-\frac{1}{m}} + \frac{1}{2}(m-1)n$.

Let ex$(n, T_2(2m)) \leq$ RHS $\leq$ ex$(n, K_2)$ + $\varepsilon\binom{n}{2} - 1$. Since ex$(n, K_2)$ = 0, and we consider the condition that $n \to \infty$, we have

$$\frac{1}{2}n^{2-\frac{1}{m}} \leq \frac{1}{2}(m-1)^{\frac{1}{m}} n^{2-\frac{1}{m}} \lesssim \frac{\varepsilon}{2}n^2$$

Which means $\frac{1}{\varepsilon} \gtrsim n^{\frac{1}{m}}$,then we have m $\gtrsim \log_{1/\varepsilon} n$.

- **Theorem 2 (Bollobás-Erdös).** $f_2(n,\varepsilon) \lesssim 2\lceil \log_{1/\varepsilon} n \rceil$.

- **Proof of Theorem 2.** Let $t = 2\lceil \log_{1/\varepsilon} n \rceil$.

We need to construct a $K_{t,t}$-free $n$-vertex graph $G$ with e$(G) > \varepsilon\binom{n}{2} - 1$.

Consider random graph $G = G(n,\varepsilon)$. i.e. agraph with $n$ vertices, where each edge is present independently with probability $\varepsilon$.

Let $X = \#K_{t,t}$'s in $G$.

So $\mathbb{E}[X] = \frac{1}{2}\binom{n}{2t}\binom{2t}{t}\varepsilon^{t^2} < n^{2t}\varepsilon^{t^2} = (n^2\varepsilon^t)^t$.

Since $t = 2\lceil \log_{1/\varepsilon} n \rceil$, we have $\varepsilon^t \leq \varepsilon^{\log_{1/\varepsilon} n} = \frac{1}{n^2}$. $\Longrightarrow \mathbb{E}[X] < (n^2\varepsilon^t)^t \leq 1$.

By deletion method, let $G'$ be obtained from $G$ by deleting one edge from each copy of $K_{t,t}$ in $G$. So $G'$ is $K_{t,t}$-free and $e(G') \geq e(G) - X$.

$\Rightarrow \mathbb{E}[e(G')] \geq \mathbb{E}[e(G) - X] = \mathbb{E}[e(G)] - \mathbb{E}[X] = \varepsilon\binom{n}{2} - 1.$

There exists a $G'$ (which is $K_{t,t}$-free) with $e(G') \geq \mathbb{E}[e(G')] > \varepsilon\binom{n}{2} - 1$.

$\Rightarrow \mathrm{ex}(n, K_{t,t}) > \varepsilon\binom{n}{2} - 1.$

$\Rightarrow f_2(n,\varepsilon) < t = 2\lceil \log_{1/\varepsilon} n \rceil.$

- **Corollary .** For $\forall\ \varepsilon > 0$, $\log_{1/\varepsilon} n \lesssim f_2(n,\varepsilon) \lesssim 2\lceil \log_{1/\varepsilon} n \rceil$.

- **Theorem 3.** For $0 < \varepsilon < \frac{1}{r(r+1)}$, $f_{r+1}(n,\varepsilon) \leq f_2\left(\left\lceil \frac{n}{r} \right\rceil, r(r+1)\varepsilon\right)$.

- **Proof of Theorem 3.** Let $t = f_2\left(\left\lceil \frac{n}{r} \right\rceil, r(r+1)\varepsilon\right) + 1$

Then, there exists a $K_{t,t}$-free $\left\lceil \frac{n}{r} \right\rceil$-vertex graph $H$ with $e(H) > r(r+1)\varepsilon\binom{n}{2} - 1$.

We want to construct a $T_{r+1}\big((r+1)t\big)$-free graph $G$ with relatively many edges.

Let $G$ be obtained from $T_r(n)$ by adding $H$ into one of its $r$ parts.

But $e(G) = e(T_r(n)) + e(H)$

$$> \mathrm{ex}(n, K_{r+1}) + r(r+1)\varepsilon\binom{\lceil \frac{n}{r} \rceil}{2} - 1$$

$$\geq \mathrm{ex}(n, K_{r+1}) + \varepsilon\binom{n}{2} - 1.$$

By define, $f_{r+1}(n,\varepsilon) < t = f_2\left(\left\lceil \frac{n}{r} \right\rceil, r(r+1)\varepsilon\right)$

$\Rightarrow f_{r+1}(n,\varepsilon) \leq f_2\left(\left\lceil \frac{n}{r} \right\rceil, r(r+1)\varepsilon\right).$

- **Exercise .** Prove $G$ is $T_{r+1}\big((r+1)t\big)$-free.

- **Corollary .** For $r \geq 1$ and $\varepsilon = \varepsilon(n) = o(1)$, $f_{r+1}(n,\varepsilon) \lesssim 2\log_{1/\varepsilon} n$.

- **Proof of Corollary .** Exercise (by theorem 2 and 3).

The best bound is obtained by Ishigami using Szemeredis regularity lemma.

- **Theorem (Ishigami).** Let $r \geq 2$ and $\varepsilon = o(1)$. Then $f_r(n,\varepsilon) \sim f_2(n,\varepsilon)$. In particular, $\log_{1/\varepsilon} n \lesssim f_r(n,\varepsilon) \lesssim 2\log_{1/\varepsilon} n$.

- **The Moon-Moser inequalities.** Let $G$ be a graph and $N_i$ be the number of copies of $K_i$ in $G$.

- **Theorem 4.** $N_3 \geq \frac{4N_2}{3}\left(\frac{N_2}{N_1} - \frac{N_1}{4}\right)$.

- **Proof of Theorem 4.** For an edge $e$, let $d(e)$ be the number of triangle containing $e$. Then

$$3N_3 = \sum_{e \in E(G)} d(e) \geq \sum_{u,v \in E(G)} (d(u) + d(v) - n) = \sum_{u \in V(G)} d^2(u) - nm$$

By Cauchy-Scharwz inequality,

$$\sum_{u\in V(G)} d^2(u) - nm \geq n\left(\frac{\sum_{u\in V(G)} d(u)}{n}\right)^2 - nm = \frac{4m^2}{n} - nm$$

- **Remark .** This implies $\mathrm{ex}(n, K_3) \leq \frac{n^2}{4}$.

- **Moon-Moser Theorem .** If $N_{s-1} \neq 0$, then

$$\frac{N_{s+1}}{N_s} \geq \frac{s^2}{s^2-1}\left(\frac{N_s}{N_{s-1}} - \frac{n}{s^2}\right)$$

In fact, we show an even stronger result, generalizing this to $r$-graphs.

- **Theorem 5.** Let $G$ be an $r$-graph, let $N_s = \#$ copies $K_s^{(r)}$ in $G$. If $N_{s-1} \neq 0$, then

$$\frac{N_{s+1}}{N_s} \geq \frac{s^2}{(s-r+1)(s+1)}\left(\frac{N_s}{N_{s-1}} - \frac{(r-1)(n-s)+s}{s^2}\right)$$

- **Proof of Theorem 5.** In this proof, we denote

(1) $e =$ clique.

(2) $\mathcal{C}_s = \{\text{all } K_s^{(r)} \text{ in } G\}$

(3) If $e \in \mathcal{C}_s$, then $d(e) = \#$ of $K_{s+1}^{(r)}$ containing $e$.

It is easy to see $\sum_{e\in\mathcal{C}_s} d(e) = (s+1)N_{s-1}$.

For $e \in \mathcal{C}_s$, denote $e_1, e_2, \cdots, e_s$ the $s$ copies of $K_{s-1}$ contained in $e$.

**Claim .** For $\forall e \in \mathcal{C}_s$, $d(e) \geq \frac{\sum_i d(e_i)-(n-s)(r-1)-s}{s-r+1}$.

**Proof of claim .** For fixed $e$, we count the number $T_e$ of pairs $(A, v)$ s.t. $|A| = s-1$, $A \subseteq e$, $v \notin e$ and $A \cup \{v\} \in \mathcal{C}_s$.

On one hand,

$T_e = \sum_{\substack{|A|=s-1\\ A\subseteq e}} \#(A, v) = \sum_i (d(e_i) - 1)$.

On the other hand, $T_e = \sum_{v\notin e} \#(A, v)$

**Property .** If vertex $u$ does not form a copy of $K_{s+1}^{(r)}$ with $e$, then there are at most $r-1$ many $e_i$'s such that $e_i \cup \{u\} \in \mathcal{C}_s$.

Why? Since $e \cup \{u\}$ is not $K_{s+1}^{(r)}$, there exists a subset $R \subseteq e$ with $|R| = r-1$ s.t. $R \cup \{u\}$ is not an edge.

Then only those $e_i$ obtained from $e$ by deleting a vertex of $R$ can be $K_s^{(r)}$.

Thus, there are at most $r-1$ such $e_i$'s.

Then for those $u \notin e$ satisfying $e \cup \{u\} \in \mathcal{C}_{s+1}$, $\#(A, u) = s$.

And for those $u \notin e$ with $e \cup \{u\} \notin \mathcal{C}_s$, by property, $\#(A, u) \leq r-1$.

Then $T_e = \sum_{v\notin e} \#(A, v) \leq d(e)s + \big(n - s - d(e)\big)(r-1)$

$\Longrightarrow d(e)s + \big(n - s - d(e)\big)(r-1) \geq T_e = \sum_i d(e_i) - \mathrm{s}$.

Proof of claim done.

By claim,

$$
\begin{aligned}
(s-r+1)(s+1)N_{s+1} &= (s-r+1)\sum_{e\in\mathcal{C}_s} d(e) \\
&\geq (s-r+1)\sum_{e\in\mathcal{C}_s} \frac{\sum_i d(e_i)-(n-s)(r-1)-s}{s-r+1}. \\
&= \sum_{e\in\mathcal{C}_s}\sum_i d(e_i) - N_s\big((n-s)(r-1)+s\big) \\
&= \sum_{f\in\mathcal{C}_{s-1}} d^2(f) - N_s\big((n-s)(r-1)+s\big) \\
&\geq N_{s-1}\left(\sum_{f\in\mathcal{C}_{s-1}} \frac{d(f)}{N_{s-1}}\right)^2 - N_s\big((n-s)(r-1)+s\big) \\
&\geq \frac{s^2N_s^2}{N_{s-1}} - N_s\big((n-s)(r-1)+s\big)
\end{aligned}
$$

Proof done.